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ON FIRST PASSAGE TIME STRUCTURE OF RANDOM WALKS

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For continuous time birth-death processes on $\{0, 1, 2, \ldots\}$, the first passage time T_n^+ from n to $n+1$ is always a mixture of $(n+1)$ independent exponential random variables. Furthermore, the first passage time $T_{0,n+1}$ from 0 to $(n+1)$ is always a sum of $(n+1)$ independent exponential random variables. The discrete time analogue, however, does not necessarily hold in spite of structural similarities. In this paper, some necessary and sufficient conditions are established under which T_n^+ and $T_{0,n+1}$ for discrete time birth-death chains become a mixture and a sum, respectively, of $(n+1)$ independent geometric random variables on $\{1, 2, \ldots\}$. The results are further extended to conditional first passage times.

birth-death processes * discrete time birth-death chains * first passage times * conditional first passage times * complete monotonicity * strong unimodality * PF_∞

0. Introduction and summary

Birth-death processes are a simple family of time reversible Markov processes of great importance, which have application to a variety of areas such as bacteriology, ecology, inventory theory, machine maintenance, congestion theory, and reliability theory. Their first passage time distributions are of corresponding interest and many papers have been published, giving insight into their structural properties, limiting behaviors and related limit theorems, see e.g., Karlin and McGregor [6], Keilson [9, 10, 11], Rösler [14], and others. In particular, it has been shown in Keilson [9, 10] that for any continuous time birth-death process on $\{0, 1, \ldots\}$, the first passage time T_n^+ from n to $(n+1)$ is always a mixture of $(n+1)$ independent exponential random variables. Furthermore, the first passage time $T_{0,n+1}$ from 0 to $(n+1)$ is always a sum of $(n+1)$ independent exponential random variables.

Of related interest, is the conditional first passage time ${}^mT_{j,n}$ $(m<j<n)$ from j to n given no visits to m. For this conditional first passage time, introduced by Keilson [11] and studied further by Sumita [18], it has been shown that ${}^mT_n^+ = {}^mT_{n,n+1}$ and ${}^mT_{m+1,n+1}$ are also a mixture and a sum, respectively, of $(n-m)$ independent exponential random variables.

Because of structural similarities between continuous time birth-death processes and discrete time birth-death chains (which we call simple random walks throughout the paper), one might expect discrete time analogues of these results. The expected analogues, however, are not always present. The purpose of this paper is to establish necessary and sufficient conditions under which T_n^+ and $T_{0,n+1}$ for simple random walks become a mixture and a sum, respectively, of $(n+1)$ independent geometric random variables on $\{1, 2, \ldots\}$. The results are further extended to conditional first passage times.

In Section 1, two theorems are first established characterizing mixtures and sums of independent, distinct geometric random variables on $\{1, 2, \ldots\}$, in terms of pole structure of the corresponding probability generating functions. Various properties on probability sequences, such as complete monotonicity, log-concavity, unimodality, and PF_∞, are also summarized which provide additional insight into structural properties of first passage times of discrete time birth-death chains. Using these results in Section 2, a necessary and sufficient condition is established under which the above discrete time analogues hold. Simpler sufficient conditions are also given. In Section 3, the conditional first passage time ${}^mT_{jn}$ is introduced. The recursion formula for probability generating functions of ${}^mT_n^+$ is derived analytically and a full probabilistic interpretation is provided. The results in Section 2 are then extended to those for conditional first passage times. The probability generating function of the downward conditional first passage time ${}^{n+1}T_{jm}$ $(m<j<n+1)$ can be easily obtained from that of ${}^mT_{j,n+1}$. This leads to the discrete counterpart of a somewhat surprising result due to Sumita [18]. It is shown that for any discrete time birth-death chains ${}^mT_{m+1,n+1} \overset{d}{=} {}^{n+1}T_{n,m}$. A stronger result is given under spatial homogeneity.

1. Characterization of mixtures and sums of n independent geometric random variables

In studying the first passage time structure of simple random walks, it is important to distinguish geometric random variables on $\{0, 1, 2, \ldots\}$ from those on $\{1, 2, \ldots\}$. We denote the former class by GM_0, and the latter by GM_1. It is clear that if $X \in GM_1$, then $(X-1) \in GM_0$.

Let $(X_j)_0^\infty$ be a sequence of independent geometric random variables in GM_1, each having a probability vector

$$f_j(m) = (1-\alpha_j)\alpha_j^{m-1}, \quad 0<\alpha_j<1, \; m = 1, 2, \ldots \tag{1.1}$$

where $\alpha_i \neq \alpha_j$ for $i \neq j$. The corresponding probability generating function (p.g.f.) is given by

$$\phi_j(u) = E(u^{X_j}) = \frac{(1-\alpha_j)u}{1-\alpha_j u}, \quad |u| < \frac{1}{\alpha_j}. \tag{1.2}$$

In this section, we consider a mixture of n independent geometric random variables Z_n where $Z_n = X_j$ with probability h_j, $j = 1, 2, \ldots n$ and a sum $Y_n = \sum_{j=1}^n X_j$. The following two theorems characterize the corresponding p.g.f.'s $E(u^{Z_n}) = \sum_{j=1}^n h_j\phi_j(u)$ and $E(u^{Y_n}) = \prod_{j=1}^n \phi_j(u)$, which play a key role throughout the paper.

Theorem 1.1. *$\gamma_n(u)$ is a p.g.f. of a mixture of n independent distinct geometric random variables in* GM_1 *if and only if the following conditions are satisfied*:

(i) $\gamma_n(1)=1$,

(ii) $$\gamma_n(u)=K_n u\cdot\frac{g_{n-1}(u)}{g_n(u)}$$

where $g_n(u)$ is a polynominal of order n with leading coefficient $(-1)^n$.

(iii) *$g_n(u)$ has n distinct real zeros on $(1,\infty)$ and the zeros of $g_{n-1}(u)$ interleave those of $g_n(u)$.*

Proof. We first assume that the conditions (i), (ii), and (iii) hold. One then sees that

$$\gamma_n(u)=K_n u\frac{\prod_{i=1}^{n-1}(a_i-u)}{\prod_{i=1}^{n}(b_i-u)} \tag{1.3}$$

where

$$1<b_1<a_1<\cdots<b_i<a_i<\cdots<a_{n-1}<b_n \tag{1.4}$$

and

$$K_n=\prod_{i=1}^{n}(b_i-1)\Big/\prod_{i=1}^{n-1}(a_i-1). \tag{1.5}$$

Equation (1.3) can be transformed into a summation formula given by

$$\gamma_n(u)=\sum_{j=1}^{n}h_j\cdot\frac{\left(1-\frac{1}{b_j}\right)u}{1-\frac{1}{b_j}u} \tag{1.6}$$

where

$$h_j=\prod_{\substack{i=1\\ i\neq j}}^{n}\left(\frac{b_i-1}{b_i-b_j}\right)\prod_{i=1}^{n-1}\left(\frac{a_i-b_j}{a_i-1}\right),\quad 1\leq j\leq n. \tag{1.7}$$

It is clear that $h_j>0$ since, from (1.4), both of the two products in (1.7) contain $(j-1)$ negative factors. Since $\gamma_n(1)=1$, one has $\sum_{\gamma=1}^{n}h_j=1$ and $\gamma_n(u)$ corresponds to a mixture of n independent geometric random variables.

Conversely, let

$$\gamma_n(u)=\sum_{j=1}^{n}h_j\cdot\frac{(1-\alpha_j)u}{1-\alpha_j u} \tag{1.8}$$

where $h_j > 0$, $\sum_{j=1}^{n} h_j = 1$ and $0 < \alpha_n < \alpha_{n-1} < \cdots < \alpha_1 < 1$. The condition (i) hold trivially. Equation (1.8) can be written as

$$\gamma_n(u) = \frac{z_{n-1}(u)}{z_n(u)} u \tag{1.9}$$

where $z_{n-1}(u)$ and $z_n(u)$ are polynomials of order $n-1$ and n respectively. Furthermore, $z_n(u)$ has n distinct zeros $1/\alpha_j$ on $(1, \infty)$. We note from (1.8) that

$$\frac{d}{du}\gamma_n(u) = \sum_{j=1}^{n} h_j \frac{(1-\alpha_j)}{(1-\alpha_j u)^2} > 0 \quad \text{for } u \neq \frac{1}{\alpha_j}.$$

Hence, $\gamma_n(u)$ is increasing between each interval $(1/\alpha_j, 1/\alpha_{j+1})$, $j = 0, 1, 2, \ldots, n$ where $1/\alpha_0 := -\infty$ and $1/(\alpha_{n+1}) := +\infty$. Since $\gamma_n(0) = 0$ and $\lim_{|u| \to \infty} \gamma_n(u) = \sum_{j=1}^{n} h_j(1 - 1/\alpha_j) < 0$, one concludes that $z_{n-1}(u)/z_n(u)$, and therefore $z_{n-1}(u)$, has $(n-1)$ distinct zeros $1/\beta_j$ satisfying

$$1 < \frac{1}{\alpha_1} < \frac{1}{\beta_1} < \frac{1}{\alpha_2} < \cdots < \frac{1}{\beta_{n-1}} < \frac{1}{\alpha_n}.$$

Hence, one has $z_n(u) = A_n \prod_{j=1}^{n} (1 - \alpha_j u)$ and $z_{n-1}(u) = A_{n-1} \prod_{j=1}^{n-1} (1 - \beta_j u)$. By letting $g_{n-1}(u) = z_{n-1}(u)/(A_{n-1} \cdot \prod_{j=1}^{n-1} \beta_j)$, $g_n(u) = z_n(u)/(A_n \cdot \prod_{j=1}^{n} \alpha_j)$, and $K_n = A_{n-1} \prod_{j=1}^{n-1} \beta_j / A_n \prod_{j-1}^{n} \alpha_j$, one has $\gamma_n(u) = K_n u \cdot g_{n-1}(u)/g_n(u)$. □

The following theorem is immediate from (1.2).

Theorem 1.2. *$\gamma_n(u)$ is a p.g.f. of a sum of n independent distinct geometric random variables in GM_1 if and only if the following conditions are satisfied.*

(i) $\gamma_n(1) = 1$,

(ii) *$\gamma_n(u) = K_n u^n / g_n(u)$ where $g_n(u)$ is a polynomial of order n with leading coefficient $(-1)^n$,*

(iii) *$g_n(u)$ has n distinct real zeros on $(1, \infty)$.*

We next review well-known classes of sequences of importance in applied probability. These classes will shed additional light in understanding structural properties of first passage times of simple random walks. Let $(a_n)_0^\infty$ be a probability sequence and define $\Delta a_n = a_n - a_{n-1}$. The higher differences $\Delta^k(a_n)$ are defined similarly.

The classes of unimodal, strongly unimodal and log-concave probability sequences, denoted by U, SU, and LCC respectively, are defined as:

Definition 1.3. (a) $(a_n)_0^\infty \in U \Leftrightarrow$ there exists a positive integer M such that $a_{n-1} \leq a_n$ for $n \leq M$ and $a_n \geq a_{n+1}$ for $n \geq M$.

(b) $(a_n)_0^\infty \in \mathrm{SU} \Leftrightarrow \left[(b_n)_0^\infty \in U \Rightarrow \left(\sum_{m=0}^{n} a_{n-m} b_m \right)_0^\infty \in U \right]$

(c) $(a_n)_0^\infty \in \mathrm{LCC} \Leftrightarrow$ there are no gaps in the support interval and $a_n^2 \geq a_{n+1} a_{n-1}$ for all n.

Of related interest are a class of Polya frequency sequences of order r and a class of totally additive sequences of order r. The former class is denoted by PF_r and the latter class by TA_r.

Definition 1.4

(a) $$(a_n)_0^\infty \in \mathrm{PF}_r \quad \Leftrightarrow \quad \begin{bmatrix} a_0 & a_1 & a_2 & & \cdots \\ 0 & a_0 & a_1 & a_2 & \cdots \\ 0 & 0 & a_0 & a_1 & \cdots \\ \vdots & \vdots & & \ddots & \ddots \end{bmatrix} \in \mathrm{TP}_r,$$

i.e. the determinant of every $k \times k$ minor of the matrix is nonnegative, $1 \leq k \leq r$.

(b) $(a_n)_0^\infty \in \mathrm{TA}_r \Leftrightarrow$ the associated random variable is a sum of r independent geometric random variables in GM_0.

The following relations exist among the classes described above. The reader is referred to Karlin [4], Keilson [10], Keilson and Gerber [12] and references thereof for more detailed discussions.

$$U \supset \mathrm{SU} = \mathrm{LCC} = \mathrm{PF}_2 \supset \mathrm{PF}_r \supset \mathrm{PF}_\infty \supset \mathrm{TA}_r. \tag{1.10}$$

Another important class is a class of completely monotone sequences denoted by CM.

Definition 1.5. $(a_n)_0^\infty \in \mathrm{CM} \Leftrightarrow (-1)^k \Delta^k (a_n) \geq 0$ for $k \geq 1$.

It can be easily seen from VII.3, Theorem 1 of Feller [1] that

$$(a_n)_0^\infty \in \mathrm{CM} \quad \Leftrightarrow \quad \text{there exists a distribution function } G(\theta)$$

$$\text{such that } \sigma(u) = \sum_{n=0}^{\infty} a_n u^n$$

$$= \int_0^1 \frac{1-\theta}{1-\theta u} \, \mathrm{d}G(\theta), |u| < 1. \tag{1.11}$$

For notational convenience, we denote the class of random variables each of which is a mixture of r independent geometric random variables in GM_0 by CM_r. We note from (1.11) the CM is the closure of $\bigcup_{r=1}^\infty \mathrm{CM}_r$ and

$$U \supset \mathrm{CM} \supset \mathrm{CM}_r. \tag{1.12}$$

For a random variable X with a probability sequence $(a_n)_0^\infty$, we write $X \in \mathrm{CM}$ or $(a_n)_0^\infty \in \mathrm{CM}$ etc., when there is no confusion.

2. First passage time structure of simple random walks

For continuous time birth-death processes on $\{0, 1, 2, \ldots\}$, the first passage time T_n^+ from n to $n+1$ is always a mixture of $(n+1)$ independent exponential random variables. Furthermore, the first passage time $T_{0,n+1}$ from 0 to $n+1$ is always a sum of $(n+1)$ independent exponential random variables, see Keilson [10]. The discrete time analogue, however, does not always hold. In this section, we establish some necessary and sufficient conditions under which T_n^+ and $T_{0,n+1}$ for simple random walks become a mixture and a sum, respectively, of $(n+1)$ independent geometric random variables in GM_1. The study of first passage time structure of random walks can be traced back to the late 50's, see e.g. Harris [2] and Karlin and McGregor [8]. An important recent paper by Whitehurst [19] provides a key tool for our analysis in this section.

Let $N(k)$ be a simple random walk on $\{0, 1, 2, \ldots\}$ governed by one step transition probability matrix $\boldsymbol{a} = (a_{mn})$ where

$$a_{mn} = \begin{cases} q_m > 0, & n = m-1, m > 1, \\ r_m > 0, & n = m, m \geqslant 0 \\ p_m > 0, & n = m+1, m \geqslant 0, \\ 0, & \text{else.} \end{cases} \quad p_m + q_m + r_m = 1, \quad p_0 + r_0 = 1 \tag{2.1}$$

Let T_n^+ be a first passage time of $N(k)$ from n to $(n+1)$ and define its probability sequence and p.g.f. by

$$\begin{aligned} s_n^+(k) &= P(T_n^+ = k), \quad k = 1, 2, \ldots, \\ \sigma_n^+(u) &= \sum_{k=1}^{\infty} s_n^+(k) u^k, \quad |u| \leqslant 1. \end{aligned} \tag{2.2}$$

It can be readily seen that

$$\sigma_0^+(u) = \frac{p_0 u}{1 - r_0 u}, \quad |u| < \frac{1}{r_0}, \tag{2.3}$$

i.e., $T_0^+ \in \mathrm{GM}_1$. To study T_n^+ for $n \geqslant 1$, we consider the lossy process $N^*(k)$ on $\{0, 1, \ldots, n+1\}$ obtained from the original chain by setting $r_{n+1} = 1$ so that the state $(n+1)$ is absorbing. Since $s_n^+(k) = p_n \cdot P(N^*(k-1) = n \mid N^*(0) = n)$, one finds after a little algebra that

$$\sigma_n^+(u) = p_n u \cdot \frac{\det(\boldsymbol{I} - u\boldsymbol{a}_{n-1}^*)}{\det(\boldsymbol{I} - u\boldsymbol{a}_n^*)}, \tag{2.4}$$

where

$$\boldsymbol{a}_n^* = \begin{bmatrix} r_0 & p_0 & 0 & \cdots & \cdots & 0 \\ q_1 & r_1 & p_1 & \ddots & & \vdots \\ 0 & q_2 & r_2 & p_2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & q_{n-1} & r_{n-1} & p_{n-1} \\ 0 & \cdots & \cdots & 0 & q_n & r_n \end{bmatrix}. \tag{2.5}$$

For notational convenience, we define $\boldsymbol{a}_n^* = 0$ for $n < 0$ and $\boldsymbol{a}_0^* = r_0$ so that (2.4) with $n = 0$ coincides with (2.3). By expanding $\det(\boldsymbol{I} - u\boldsymbol{a}_n^*)$, the following proposition holds.

Proposition 2.1

$$\text{(a)} \qquad \sigma_n^+(u) = \frac{p_n u}{1 - r_n u - q_n u \sigma_{n-1}^+(u)}, \quad n \geq 1,$$

with $\sigma_0^+(u) = p_0 u/(1 - r_0 u)$, $|u| \leq 1$,

$$\text{(b)} \qquad E(T_n^+) = \frac{1}{p_n}\{1 + q_n E(T_{n-1}^+)\}, \quad n \geq 1$$

with $E(T_0^+) = 1/p_0$

Proof. We note from (2.5) that

$$\det(\boldsymbol{I} - u\boldsymbol{a}_n^*) = (1 - u r_n)\det(\boldsymbol{I} - u\boldsymbol{a}_{n-1}^*) - q_n p_{n-1} u^2 \det(\boldsymbol{I} - u\boldsymbol{a}_{n-2}^*). \quad \square \tag{2.6}$$

Substituting (2.6) into (2.4) and then dividing both numerator and denominator by $\det(\boldsymbol{I} - u\boldsymbol{a}_{n-1}^*)$, statement (a) follows. Statement (b) can be derived by differentiating $\sigma_n^+(u)$ in (a) at $u = 1$.

As for the continuous time case (cf. (5.2.3) of Keilson [10]), using the potential coefficients $\pi_n = \prod_{j=0}^{n-1} p_j/q_{j+1}$, $n \geq 1$, $\pi_0 = 1$, one has

$$E(T_n^+) = \frac{1}{p_n \pi_n} \sum_{j=0}^{n} \pi_j, \quad n \geq 0. \tag{2.7}$$

It can be readily seen from Proposition 2.1(a) that

$$\sigma_n^+(u) = p_n u + r_n u \sigma_n^+(u) + q_n u \sigma_{n-1}^+(u)\sigma_n^+(u). \tag{2.8}$$

Equivalently, one has,

$$s_n^+(k) = p_n \delta_{k,1} + r_n s_n^+(k-1) + q_n P(T_{n-1}^+ + T_n^+ = k-1) \tag{2.9}$$

where $\delta_{k,1} = 1$ for $k = 1$ and $\delta_{k,1} = 0$ otherwise. The probabilistic meaning of (2.9) is clear.

Using the result of Whitehurst [19], we next establish a necessary and sufficient condition under which T_n^+ is a mixture of $n+1$ independent geometric random variables in GM_1.

Theorem 2.2. *T_n^+ is a mixture of $(n+1)$ independent, distinct geometric random variables in* GM_1 *(i.e., $(T_n^+ - 1) \in \mathrm{CM}_{n+1}$) if and only if* $\det \boldsymbol{a}_m^* > 0$ *for $m = 1, 2, \ldots, n$, where $\boldsymbol{a}_n^*$ is as given in* (2.5).

Proof. Let $Q_0(x) = 1$ and define

$$Q_n(x) = (-1)^n \frac{\det(\boldsymbol{a}_{n-1}^* - x\boldsymbol{I})}{p_0 p_1 \cdots p_{n-1}}, \quad n \geq 1. \tag{2.10}$$

By expanding the determinant, one then easily sees that

$$xQ_n(x) = p_n Q_{n+1}(x) + r_n Q_n(x) + q_n Q_{n-1}(x), \quad n \geq 1. \tag{2.11}$$

We note that the recursion formula (2.11) coincides with (1.1) of Whitehurst [19] and defines an orthogonal polynomial system on the interval $[-1, 1]$. Hence the zeros of $Q_n(x)$ are in the interval $(-1, 1)$ and the zeros of $Q_n(x)$ and $Q_{n+1}(x)$ interleave (see Whitehurst [19] and references thereof). From (2.4) one observes that $\sigma_n^+(u) = Q_n(u^{-1})/Q_{n+1}(u^{-1})$, $n \geq 0$ so that

$$\sigma_n^+(u) = u\frac{z_n(u)}{z_{n+1}(u)}, \quad z_n(u) = u^n Q_n(u^{-1}), \qquad n \geq 0. \tag{2.12}$$

Clearly $z_n(u)$ has its zeros in $(1, \infty)$ if and only if $Q_n(u^{-1})$ has its zeros in $(1, \infty)$ or equivalently $Q_n(u)$ has zeros in $(0, 1)$. This holds by induction if and only if $(-1)^m Q_m(0) > 0$, $0 \leq m \leq n$, i.e., $\det \boldsymbol{a}_m^* > 0$ for $0 \leq m \leq n$. The theorem now follows from Theorem 1.1. □

Remark 2.3. As the referee pointed out, the condition $\det \boldsymbol{a}_m^* > 0$, $0 \leq m \leq n$ can be expressed in an alternative form in terms of continuous fractions, see Theorem 2.3 of Whitehurst [19].

It should be noted that if $\det \boldsymbol{a}_m^* > 0$, $m = 1, 2, \ldots, n$, then zeros of $z_{n+1}(u)$ interleave those of $z_n(u)$ on $(1, \infty)$, and can be calculated straightforwardly via the bisection method. Those zeros enable one to evaluate $(s_n^+(k))_{k=1}^{\infty}$ explicitly by converting $\sigma_n^+(u)$ in (2.12) to a summation formula using (1.6) and (1.7). Numerical difficulty may arise when the zeros of $z_{n+1}(u)$ become very close to each other.

When $\det \boldsymbol{a}_m^* > 0$, $1 \leq m < n$, and $\det \boldsymbol{a}_n^* < 0$, one of zeros of $z_{n+1}(u)$ becomes negative. Correspondingly, $\sigma_n^+(u)$ takes the form

$$\sigma_n^+(u) = \sum_{j=1}^{n} h_j \frac{(1-\theta_j)u}{1-\theta_j u} + h^* \frac{(1+\theta^*)u}{1+\theta^* u} \quad \text{where } \theta_j > 0,\ \theta^* > 0,$$

destroying the property $(T_n^+ - 1) \in \mathrm{CM}_{n+1}$.

The next corollary is immediate from Theorem 2.2.

Corollary 2.4. *If* $(T_n^+ - 1) \in \mathrm{CM}_{n+1}$, *then* $(T_r^+ - 1) \in \mathrm{CM}_{r+1}$, $r = 0, 1, 2, \ldots, n-1$.

Let a_n be a leading coefficient of $z_n(u)$ and define $b_n = (-1)^n a_n$. One then sees from (2.11) and (2.12) that

$$b_{n+1} = \frac{1}{p_n}(r_n b_n - q_n b_{n-1}), \quad n \geq 1, \tag{2.13}$$

where $b_0 = 1$ and $b_1 = r_0/p_0$. Since $a_{n+1} = (-1)^{n+1}(\det \boldsymbol{a}_n)/p_0 \cdots p_n$, $n \geq 0$, it follows that the necessary and sufficient condition is equivalent to $b_m > 0$, $0 \leq m \leq n+1$. This observation leads to a simpler sufficient condition.

Theorem 2.5. *If* $r_m \geq \frac{1}{2}$, $0 \leq m \leq n$, *then* $(T_m^+ - 1) \in \mathrm{CM}_{m+1}$, $0 \leq m \leq n$.

Proof. Since $r_m + q_m + p_m = 1$, the condition $r_m \geq \frac{1}{2}$ is equivalent to $r_m \geq q_m + p_m$. We show by induction that b_m generated by (2.13) is nondecreasing under this condition. One has trivially $b_0 = 1 \leq b_1 = r_0/p_0$. Suppose $b_0 \leq b_1 \leq \cdots \leq b_m$. Then from (2.13) one sees that

$$b_{m+1} - b_m = \frac{1}{p_m}\{(r_m - p_m)b_m - q_m b_{m-1}\} \geq \frac{1}{p_m}(r_m - p_m - q_m)b_m \geq 0.$$

Hence $0 < b_0 \leq b_m$ for $0 \leq m \leq n+1$, completing the proof. □

When the spatial homogeneity is present, the necessary and sufficient condition can be simplified.

Theorem 2.6. *Let* $p_n = p$, $n \geq 0$, $r_0 = 1 - p$, $r_n = r$, $n \geq 1$, *and* $q_n = q$, $n \geq 1$ *where* p, q, $r > 0$ *and* $p + q + r = 1$. *Then* $(T_n^+ - 1) \in \mathrm{CM}_{n+1}$ *for* $n = 0, 1, 2, \ldots$ *if and only if* $r^2 \geq 4pq$.

Proof. The theorem follows from the discussion of homogeneous Jacobi matrices given in pp. 112–117 of Karlin [4]. □

Remark 2.7. A difference, that may come to one's mind first, between continuous time birth–death processes and discrete time birth–death chains is the TP property of transition matrices. It has been shown by Karlin and McGregor [5, 7] that for any continuous time birth–death process its transition probability matrix $\boldsymbol{P}(t)$ belongs to TP_∞ and hence to TP_2. For a discrete time birth–death chain, its one-step transition probability matrix $\boldsymbol{a}$ may not be even TP_2. As we have seen in Theorem 2.2 the condition $\boldsymbol{a} \in \mathrm{STP}_\infty$ assures $(T_n^+ - 1) \in \mathrm{CM}_{n+1}$ for all n, $n \geq 0$ where the class STP_r is obtained by replacing the nonnegativity requirement of $k \times k$ minors $1 \leq k \leq r$ for TP_r by the positivity requirement. The condition $\boldsymbol{a} \in \mathrm{STP}_2$, however, is not sufficient. For the case of spatial homogeneity as in Theorem 2.6, the condition $\boldsymbol{a} \in \mathrm{STP}_2$ is equivalent to $r^2 > pq$. Hence by setting $r = 2/(2+\sqrt{2})$ and $p = q = 1/(2+\sqrt{2})$ so that $r^2 = 2pq$, one has $\boldsymbol{a} \in \mathrm{STP}_2$, but the condition $r^2 \geq 4pq$ of Theorem 2.6 is violated.

We now turn our attention to the first passage time T_{0n} from 0 to n. The conditions of Theorem 2.2 assure that $(T_{0,n+1} - (n+1)) \in \mathrm{TA}_{n+1}$ as we prove next.

Theorem 2.8. $T_{0,n+1}$ *is a sum of* $(n+1)$ *independent, distinct geometric random variables in* GM_1 (i.e., $(T_{0,n+1} - (n+1)) \in \mathrm{TA}_{n+1}$) *if and only if* $\det \boldsymbol{a}_m^* > 0$ *for* $m = 1, 2, \ldots, n$.

Proof. It is easy to see from (2.12) that $\sigma_{0,n+1}(u)=E(u^{T_{0,n+1}})=u^{n+1}/z_{n+1}(u)$. If $\det \boldsymbol{a}_m^*>0$, $m=1,2,\ldots,n$, then $z_{n+1}(u)$ has $(n+1)$ distinct zeros on $(1,\infty)$ as shown in the proof of Theorem 2.2. Hence from Theorem 1.2, one has $(T_{0,n+1}-(n+1))\in \mathrm{TA}_{n+1}$. Conversely, if $(T_{0,n+1}-(n+1))\in \mathrm{TA}_{n+1}$, one sees from Theorem 1.2 that $z_{n+1}(u)$ has $(n+1)$ distinct zeros on $(1,\infty)$. Since the zeros of $z_n(u)$ interleave those of $z_{n+1}(u)$, one obtains from Theorem 1.1 $(T_n^+-1)\in \mathrm{CM}_{n+1}$. This then implies from Theorem 2.2 that $\det \boldsymbol{a}_m^*>0$, $m=1,2,\ldots,n$, completing the proof. □

We note from (1.10) that $(T_{0,n+1}-(n+1))\in \mathrm{TA}_{n+1}$ implies the log-concavity and the strong unimodality of $T_{0,n+1}$, since the PF_2 property is closed under sifting.

Remark 2.9. It can be readily seen from Theorem 2.2 and Theorem 2.8, that $(T_n^+-1)\in \mathrm{CM}_{n+1}$ if and only if $(T_{0,n+1}-(n+1))\in \mathrm{TA}_{n+1}$. Hence, the results in parallel to Corollary 2.4 and Theorem 2.6 hold immediately for $T_{0,n+1}$.

As stated in Remark 2.7, the condition $\boldsymbol{a}\in \mathrm{STP}_2$ does not imply $(T_{0,n+1}-(n+1))\in \mathrm{TA}_{n+1}$. One can show, however, that $\boldsymbol{a}\in \mathrm{TP}_2$ is sufficient for $T_{0,n+1}\in \mathrm{PF}_2=\mathrm{LCC}=\mathrm{SU}$ as we prove next.

Theorem 2.10. *Let* $\boldsymbol{a}\in \mathrm{TP}_2$. *Then* $T_{0,n+1}\in \mathrm{PF}_2$.

Proof. Let

$$\boldsymbol{a}_{n+1}=\begin{bmatrix} r_0 & p_0 & 0 & \cdots & 0 \\ q_1 & r_1 & p_1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & q_n & r_n & p_n \\ 0 & \cdots & 0 & 0 & 1 \end{bmatrix}.$$

The matrix $\boldsymbol{a}_{n+1}$ governs the lossy process $N^*(k)$ given in the beginning of this section. Clearly, $\boldsymbol{a}\in \mathrm{TP}_2$ implies $\boldsymbol{a}_{n+1}\in \mathrm{TP}_2$ for $n\geqslant 0$. Let $\boldsymbol{p}^*(k)=(p_1^*(k),\ldots,p_{n+1}^*(k))$ where $p_m^*(k)=p(N^*(k)=m\mid N^*(0)=0)$. One easily shows that

$$s_{0,n+1}(k)=p(T_{0,n+1}=k)=p_n p_n^*(k-1). \tag{2.1}$$

It has been shown by Keilson and Sumita [13] that, since $\boldsymbol{a}_{n+1}\in \mathrm{TP}_2$, the local ordering $\boldsymbol{p}^*(k)<^l\boldsymbol{p}^*(k+1)$ holds, i.e., $p_m^*(k)/p_m^*(k+1)$ is nonincreasing in m. Hence if we define

$$\boldsymbol{P}^*=\begin{bmatrix} \boldsymbol{p}^*(0)^T \\ \boldsymbol{p}^*(1)^T \\ \vdots \end{bmatrix}$$

then $P^* \in \mathrm{TP}_2$. This in turn implies that $P^{*T} \in \mathrm{TP}_2$ and one has

$$\frac{p^*_{n-1}(k)}{p^*_n(k)} \quad \text{is nonincreasing in } k,\ k \geq n. \tag{2.15}$$

From (2.14), one observes that

$$\frac{s_{0,n+1}(k+1)}{s_{0,n+1}(k)} = \frac{p^*_n(k)}{p^*_n(k-1)} = \frac{r_n p^*_n(k-1) + p_{n-1} p^*_{n-1}(k-1)}{p^*_n(k-1)}$$
$$= r_n + p_{n-1}\frac{p^*_{n-1}(k-1)}{p^*_n(k-1)}.$$

Hence from (2.15), $s_{0,n+1}(k+1)/s_{0,n+1}(k)$ is nonincreasing for $k \geq n+1$, and therefore $(s_{0,n+1}(k))_{k=0}^{\infty} \in \mathrm{LCC} = \mathrm{PF}_2$. □

Theorem 2.10 is implicit in the results of Karlin [3] for more general Markov chains. Similar results for downward first passage times hold straightforwardly and are omitted here.

3. Conditional first passage time structure of simple random walks

Let ${}^mT_{jn}$ be a conditional first passage time of $N(k)$ from j to n given no visits to m where $m<j<n$. For notational convenience, we write ${}^mT_{n,n+1} = {}^mT^+_n$. These conditional first passage times for continuous time birth-death processes are first introduced by Keilson [11] and their structural properties are further studied by Sumita [18]. In this section, we extend the results of Section 2 to those for conditional first passage times. We also establish discrete analogue of some results in Sumita [18].

Let $N(k)$ be a birth-death chain in discrete time described in Section 2. We define the probability sequence and the p.g.f. of ${}^mT^+_n$ by

$${}^ms^+_n(k) = P({}^mT^+_n = k); \qquad {}^m\sigma^+_n(u) = \sum_{k=1}^{\infty} {}^ms^+_n(k)u^k, \quad |u| \leq 1. \tag{3.1}$$

It can be readily seen that

$${}^m\sigma^+_{m+1}(u) = \frac{(1-r_{m+1})u}{1-r_{m+1}u}, \quad |u| < \frac{1}{r_{m+1}}. \tag{3.2}$$

It should be noted that (2.4) remains valid even when $q_0 > 0$. In such a case, $\sigma^+_n(s)$ corresponds to dishonest distribution. Thus the p.g.f. of ${}^mT^+_n$ can be found by shifting the states and normalization. Namely one has

$${}^m\sigma^+_n(u) = {}^mK^+_n u \frac{\det(I - ua^*_{m+1,n-1})}{\det(I - ua^*_{m+1,n})}, \quad m+1 \leq n,\ |u| \leq 1, \tag{3.3}$$

where $a^*_{m+1,j}=0$ for $j<m+1$,

$$a^*_{m+1,n}=\begin{bmatrix} r_{m+1} & p_{m+1} & 0 & \cdots & \cdots & 0 \\ q_{m+2} & r_{m+2} & p_{m+2} & & & \vdots \\ 0 & q_{m+3} & r_{m+3} & p_{m+3} & & 0 \\ \vdots & & \ddots & q_{n-1} & r_{n-1} & p_{n-1} \\ 0 & \cdots & \cdots & 0 & q_n & r_n \end{bmatrix}, \quad n\geq m+1, \tag{3.4}$$

and

$$^mK^+_n=\frac{\det(I-a^*_{m+1,n})}{\det(I-a^*_{m+1,n-1})}. \tag{3.5}$$

By expanding the determinant in the denominator, (3.3) leads to the recursive formula

$$^m\sigma^+_n(u)=\frac{^mK^+_n u}{1-r_n u-(p_{n-1}q_n u\,^m\sigma^+_{n-1}(u)/^mK^+_{n-1})}, \quad n>m+1,\ |u|\leq 1, \tag{3.6}$$

starting with $^m\sigma^+_{m+1}(u)$ of (3.2). The normalization constants $^mK^+_n$ are related to ruin probabilities as we show next.

Lemma 3.1. *Let $^m\theta^+_n$ be the ruin probability that the chain $N(k)$ reaches $(n+1)$ before m starting at n, $n\geq m+1$. Then $^m\theta^+_n=p_n/^mK^+_m$, $n\geq m+1$.*

Proof. One easily sees from probabilistic reasoning that

$$^m\theta^+_n=p_n+r_n\,^m\theta^+_n+q_n\,^m\theta^+_{n-1}\,^m\theta^+_n, \quad n\geq m+2$$

where $^m\theta^+_{m+1}=p_{m+1}/(p_{m+1}+q_{m+1})$. Hence,

$$^m\theta^+_n=\frac{p_n}{p_n+q_n(1-\,^m\theta^+_{n-1})}, \quad n\geq m+2. \tag{3.7}$$

On the other hand, by setting $u=1$ in (3.6) one finds that

$$\frac{p_n}{^mK^+_n}=\frac{p_n}{p_n+q_n\left(1-\dfrac{p_{n-1}}{^mK^+_{n-1}}\right)}, \quad n\geq m+2 \tag{3.8}$$

with $p_{m+1}/^mK^+_{m+1}=p_{m+1}/(p_{m+1}+q_{m+1})$. Hence $^m\theta^+_n$ and $p_n/^mK^+_n$ satisfy the same recursion formula including its starting value and the lemma follows. □

Lemma 3.1 immediately leads to the following theorem.

Theorem 3.2

(a) $$ {}^m\sigma_n^+(u) = \frac{(p_n/{}^m\theta_n^+)u}{1 - r_n u - q_n {}^m\theta_{n-1}^+ u\, {}^m\sigma_{n-1}^+(u)}, \quad n \geq m+2,\ |u| \leq 1, $$

starting with ${}^m\sigma_{m+1}^+(u)$ *of* (3.2).

(b) $$ E({}^mT_n^+) = \frac{{}^m\theta_n^+}{p_n}(1 + q_n {}^m\theta_{n-1}^+ E({}^mT_{n-1}^+)), \quad n \geq m+1, $$

with $E({}^mT_{m+1}^+) = \dfrac{1}{1 - r_{m+1}}$.

Proof. Statement (a) follows directly from Lemma 3.1 and (3.6). Statement (b) is obtained by differentiating ${}^m\sigma_n^+(u)$ in (a) at $u = 1$. □

The recursion formula for ${}^m\sigma_n^+(u)$ in Theorem 3.2(a) can be written as

$$ {}^m\sigma_n^+(u) = \frac{p_n}{{}^m\theta_n^+} u + r_n u\, {}^m\sigma_n^+(u) + q_n {}^m\theta_{n-1}^+ u\, {}^m\sigma_{n-1}^+(u)\, {}^m\sigma_n^+(u). \tag{3.9} $$

We first note that $p_n/{}^m\theta_n^+$ is the conditional probability that the chain goes up to $(n+1)$ from n in one transition given that the chain does not visit m. Similarly, $q_n {}^m\theta_{n-1}^+$ is the probability that the chain goes down to $(n-1)$ from n and comes back to n without visiting m. The probabilistic meaning of (3.9) is now clear.

Following the line of the proof of Theorem 2.2 and Theorem 2.8, one establishes the next theorem.

Theorem 3.3. *The next three statements are equivalent*:

(a) det $a_{m+1,k}^* > 0$ *for* $k = m+1, m+2, \ldots, n$ *where* $a_{m+1,k}^*$ *is given in* (3.4).

(b) ${}^mT_n^+$ *is a mixture of* $(n-m)$ *independent, distinct geometric random variables in* GM_1, i.e., $({}^mT_n^+ - 1) \in \mathrm{CM}_{n-m}$.

(c) ${}^mT_{m+1,n+1}$ *is a sum of* $(n-m)$ *independent, distinct geometric random variables in* GM_1, i.e., $({}^mT_{m+1,n+1} - (n-m)) \in \mathrm{TA}_{n-m}$.

As before, simpler sufficient condition is available.

Theorem 3.4. *The statements* (a), (b), *and* (c) *of Theorem* 3.3 *hold if* $r_j \geq \frac{1}{2}$ *for* $j = m+1, \ldots, n$.

Proof. The theorem can be proven by an argument similar to the proof of Theorem 2.5 since $(p_{k+1}/{}^m\theta_{k+1}^+) + q_{k+1} {}^m\theta_k^+ = p_{k+1} + q_{k+1}$. □

By reversing the states, the p.g.f. of the downward first passage time ${}^{n+1}T_{j+1}^-$ $(j < n)$ from $(j+1)$ to j given no visits to $(n+1)$ can be easily found from (3.3).

One has

$$ {}^{n+1}\sigma_{j+1}^{-}(u) = {}^{n+1}K_{j+1}^{-} \cdot u \cdot \frac{\det(\boldsymbol{I} - u\boldsymbol{a}_{j+2,n}^{*})}{\det(\boldsymbol{I} - u\boldsymbol{a}_{j+1,n}^{*})}. \tag{3.10} $$

The results similar to Theorem 3.2, 3.3, and 3.4 can then be shown. (3.10) also enables one to establish a discrete analogue of a somewhat surprising result for continuous time birth-death processes given by Sumita [18].

Theorem 3.6. (a) *For any* $r_j > 0$, $p_j > 0$ $(j \geqslant 0)$, *and* $q_j > 0$ $(j \geqslant 1)$,

$$ {}^{m}T_{m+1,n+1} \overset{\mathrm{d}}{=} {}^{n+1}T_{nm},\ m < n. $$

(b) *If* $r_j = r$, $p_j = p$, *and* $q_j = q$ *for* $m < j < n+1$, *then*

$$ {}^{m}T_{m+k,n+1} \overset{\mathrm{d}}{=} {}^{n+1}T_{n+1-k,m}. $$

Proof. Since ${}^{m}\sigma_{m+1,n+1}(u) = \prod_{k=m+1}^{n} {}^{m}\sigma_{k}^{+}(u)$ and ${}^{n+1}\sigma_{nm}(u) = \prod_{k=m+1}^{n} {}^{n+1}\sigma_{k}^{-}(u)$, one sees from (3.3) and (3.10) that

$$ {}^{m}\sigma_{m+1,n+1}(u) = u^{n-m} \cdot \frac{\det(\boldsymbol{I} - \boldsymbol{a}_{m+1,n}^{*})}{\det(\boldsymbol{I} - u\boldsymbol{a}_{m+1,n}^{*})} = {}^{n+1}\sigma_{nm}(u), $$

proving (a). To prove (b), we note from (3.3) and (3.10) that

$$ {}^{m}\sigma_{m+k,n+1}(u) = C \cdot u^{n+2-m-k} \frac{\det(\boldsymbol{I} - u\boldsymbol{a}_{m+1,m+k-1}^{*})}{\det(\boldsymbol{I} - u\boldsymbol{a}_{m+1,n}^{*})} $$

and

$$ {}^{n+1}\sigma_{n+1-k,m}(u) = C' \cdot u^{n+2-m-k} \frac{\det(\boldsymbol{I} - u\boldsymbol{a}_{n+2-k,n}^{*})}{\det(\boldsymbol{I} - u\boldsymbol{a}_{m+1,n}^{*})}. $$

Under the condition given, it can be readily seen that $\det(\boldsymbol{I} - u\boldsymbol{a}_{m+1,m+k-1}^{*}) = \det(\boldsymbol{I} - u\boldsymbol{a}_{n+2-k,n}^{*})$. Hence, ${}^{m}\sigma_{m+k,n+1}(u) = {}^{n+1}\sigma_{n+1-k,m}(u)$, proving the theorem. □

Some special cases of Theorem 3.6(b) in the context of the gambler's ruin problem have been discussed in Samuels [15], Seneta [16], and Stern [17].

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